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On the Asymptotic Approximation with Bivariate Operators of Bleimann, Butzer, and Hahn

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The concern of this paper is a recent generalization $L_n(f(t_1, t_2); x, y)$ for the operators of Bleimann, Butzer, and Hahn in two variables which is distinct from a tensor product. We present the complete asymptotic expansion for the operators L_n as n tends to infinity. The result is in a form convenient for applications. All coefficients of n^{-k} ($k=1, 2, ...$) are calculated explicitly in terms of Stirling numbers of the first and second kind. As a special case we obtain a Voronovskaja-type theorem for the operators L_n. The result for the one-dimensional case was previously derived by the author. © 1999 Academic Press

1. INTRODUCTION

In 1980 Bleimann, Butzer, and Hahn [10] introduced a sequence of positive linear operators $L_n^{[1]}$ defined for any real function f on the interval $[0, \infty)$ by

$$(L_n^{[1]}f)(x) = (1+x)^{-n} \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n-k+1}\right) x^k \qquad (n \in \mathbb{N}). \tag{1}$$

Throughout the paper we briefly denote them by BBH operators.

Bleimann, Butzer, and Hahn proved that, for bounded $f \in C[0, \infty)$, $L_n^{[1]} f \to f$ as $n \to \infty$ pointwise on $[0, \infty)$, the convergence being uniform on each compact subset of $[0, \infty)$. Furthermore, they found a rate of convergence by estimating $|L_n^{[1]}(f(t); x) - f(x)|$ in terms of the second modulus of continuity of f, where f is assumed to be bounded and uniformly continuous on $[0, \infty)$. For a growth condition on f which ensures pointwise

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convergence of $L_n^{[1]} f$ as $n \to \infty$ see [14, Theorem 2.1]. Several authors [26, 18, 22, 12, 13, 11, 14–16, 8, 19] studied the operators $L_n^{[1]}$ in the following (see also [9, pp. 306–310, 318]).

Totik [26, Eq. (2.6), where the factor 2^{-1} is absent] and, later independently, Mercer [22] derived the Voronovskaja-type theorem

$$\lim_{n \to \infty} n((L_n^{[1]} f)(x) - f(x)) = \frac{x(1+x)^2}{2} f''(x) \tag{2}$$

for all $f \in C^2[0, \infty)$ with $f(x) = O(x)(x \to \infty)$.

The author [2] extended this result by giving the complete asymptotic expansion for the BBH operators in the form

$$(L_n^{[1]} f)(x) \sim f(x) + \sum_{k=1}^{\infty} c_k(f; x)(n+1)^{-k} \qquad (n \to \infty) \tag{3}$$

for every function f on $[0, \infty)$ satisfying $f(x) = O(x)(x \to \infty)$ and possessing all derivatives in x. Formula (3) means that

$$(L_n^{[1]} f)(x) = f(x) + \sum_{k=1}^{m} c_k(f; x)(n+1)^{-k} + o(n^{-m}) \qquad (n \to \infty)$$

for all $m \in \mathbb{N}$. The Voronovskaja-type result (2) is the special case $m = 1$.

We remark that in [1, 3–5] the author gave the analogous results for the Meyer–König and Zeller operators, for the Kantorovich polynomials, and the Stancu beta operators, respectively.

Recently, Adell, de la Cal and Miguel [7] exhibited a bivariate version of the BBH operators as follows.

Set $\Delta := \{(x, y) \in \mathbb{R}^2 \mid x \geqslant 0, y \geqslant 0, xy < 1\}$ and define, for $(x, y) \in \Delta$, $n \in \mathbb{N}$ and any real function f on Δ

$$\begin{aligned}(L_n f)(x, y) &\equiv L_n(f(t_1, t_2); x, y)\\ &= \sum_{k=0}^{n} \sum_{\ell=0}^{n-k} f\left(\frac{k}{n-k+1}, \frac{\ell}{n-\ell+1}\right)\binom{n}{k, \ell}\\ &\quad \times \left(\frac{x}{1+x}\right)^k \left(\frac{y}{1+y}\right)^\ell \left(\frac{1-xy}{(1+x)(1+y)}\right)^{n-k-\ell} \qquad (n \in \mathbb{N})\end{aligned} \tag{4}$$

with the multinomial coefficient $\binom{n}{k, \ell} = n!/(k!\,\ell!\,(n-k-\ell)!)$. Note, that this two-dimensional analogue of the BBH operators is distinct from a tensor product.

The purpose of this paper is to derive the complete asymptotic expansion for these operators in the form

$$(L_n f)(x, y) \sim f(x, y) + \sum_{k=1}^{\infty} c_k(f; x, y)(n+1)^{-k} \qquad (n \to \infty) \tag{5}$$

for every bounded function f on Δ which possesses all derivatives in (x, y).

As special case Eq. (5) contains the Voronovskaja-type formula

$$\begin{aligned}\lim_{n\to\infty} n((L_n f)(x, y) - f(x, y)) = {}& \frac{x(1+x)^2}{2} \frac{\partial^2}{\partial x^2} f(x, y) \\ & - xy(1+x)(1+y) \frac{\partial^2}{\partial x\, \partial y} f(x, y) \\ & + \frac{y(1+y)^2}{2} \frac{\partial^2}{\partial y^2} f(x, y).\end{aligned} \tag{6}$$

All coefficients $c_k(f; x, y)$ $(k = 1, 2, ...)$ are calculated explicitly in terms of Stirling numbers of the first and second kind.

While the proof in [2] is based on the observation that the operators $L_n^{[1]}$ are intimately related to the Bernstein operators B_n by a rational transformation we use in this paper a completely other method. Our proofs are self-contained and do not use any properties of the Bernstein operators.

First we investigate the moments for the operators L_n. Then we present an extension of a general approximation theorem due to Sikkema [24, 25] into the bivariate case. Finally, we show that the operators L_n satisfy the assumptions of this theorem in order to obtain the complete asymptotic expansion (5).

The paper is organized as follows. In the next section we present the main results. Section 3 is devoted to auxiliary results and the last section contains the proofs.

2. THE MAIN RESULT

For $r \in \mathbb{N}$ and fixed $(x, y) \in \mathbb{R}^2$, let $K^{[2r]}(x, y)$ be the class of all functions $f: \mathbb{R}^2 \to \mathbb{R}$ which are bounded on each bounded subset of $\mathbb{R}^2$ with $f(t_1, t_2) = O((t_1^2 + t_2^2)^r)$ as $(t_1^2 + t_2^2) \to \infty$ and such that f and all its partial derivatives of order $\leqslant 2r$ are continuous in (x, y). Now we are in position to formulate our main result.

THEOREM 1. *Let $r \in \mathbb{N}$ and $(x, y) \in \Delta$. Then, for each $f \in K^{[2r]}(x, y)$, the bivariate BBH operators possess the asymptotic expansion*

$$(L_n f)(x, y) = f(x, y) + \sum_{k=1}^{r} c_k(f; x, y)(n+1)^{-k} + o(n^{-r}) \qquad (n \to \infty), \tag{7}$$

where the coefficients are given by

$$c_k(f; x, y) = \sum_{s=2}^{2k} \frac{(-1)^{k+s}}{s!} \sum_{P+Q=s} \binom{s}{P,\, Q} \frac{\partial^s}{\partial x^P \partial y^Q} f(x, y) \times (1+x)^P (1+y)^Q H_k(P, Q)$$

and H_k is defined as

$$\begin{aligned} H_k(P, Q) = \sum_{j=0}^{k} \sum_{\mu+\lambda \leqslant j} \sum_{m=0}^{\mu} \sum_{\ell=0}^{\lambda} & (1+x)^m (1+y)^\ell (1-xy)^{j-m-\ell} \\ & \times \sum_{p=1}^{P} \sum_{q=1}^{Q} (-1)^{p+q} \binom{P}{p} \binom{Q}{q} S_{p-1+m}^{p-1} \sigma_{p-1+\mu}^{p-1+m} \\ & \times S_{q-1+\ell}^{q-1} \sigma_{q-1+\lambda}^{q-1+\ell} S_{p+q-1+j}^{p+q-1+\mu+\lambda} \sigma_{p+q-1+k}^{p+q-1+j} \\ & + \sum_{m=0}^{k} (1+x)^m \sum_{p=1}^{P} (-1)^p \binom{P}{p} S_{p-1+m}^{p-1} \sigma_{p-1+k}^{p-1+m} \\ & + \sum_{\ell=0}^{k} (1+y)^\ell \sum_{q=1}^{Q} (-1)^q \binom{Q}{q} S_{q-1+\ell}^{q-1} \sigma_{q-1+k}^{q-1+\ell}. \end{aligned} \tag{8}$$

Note that the values of Stirling numbers can readily be computed by simple recurrences or can be found in the literature. Also they are available with the aid of computer algebra software. Therefore, it is easily possible to calculate explicit expressions for the coefficients $c_k(f; x, y)$.

As an immediate consequence of Theorem 1 we obtain the abovementioned Voronovskaja-type formula for the bivariate BBH operators.

COROLLARY 1. *For $(x, y) \in \Delta$ and $f \in K^{[2]}(x, y)$, we have*

$$\begin{aligned} \lim_{n \to \infty} n((L_n f)(x, y) - f(x, y)) = & \frac{x(1+x)^2}{2} \frac{\partial^2}{\partial x^2} f(x, y) \\ & - xy(1+x)(1+y) \frac{\partial^2}{\partial x \, \partial y} f(x, y) \\ & + \frac{y(1+y)^2}{2} \frac{\partial^2}{\partial y^2} f(x, y). \end{aligned} \tag{9}$$

As a further corollary of Theorem 1 we can deduce the complete asymptotic expansion for the BBH operators in the univariate case.

COROLLARY 2. *Let $f\colon [0, \infty) \to \mathbb{R}$ be bounded and admitting derivatives of sufficiently high order at $x \in [0, \infty)$. Then, the univariate BBH operators possess the complete asymptotic expansion*

$$(L_n^{[1]} f)(x) \sim f(x) + \sum_{k=1}^{\infty} a_k(f; x)(n+1)^{-k} \qquad (n \to \infty), \tag{10}$$

where the coefficients $a_k(f; x)$ are given by

$$a_k(f; x) = \sum_{s=2}^{2k} \frac{(-1)^{k+s}}{s!} f^{(s)}(x)(1+x)^s \\ \times \sum_{m=0}^{k} (1+x)^m \sum_{p=1}^{s} (-1)^p \binom{s}{p} S_{p-1+m}^{p-1} \sigma_{p-1+k}^{p-1+m}. \tag{11}$$

Formula (11) simplifies and corrects a previous result [2, Theorem 1].

3. AUXILIARY RESULTS

First, we note a general property of the operators L_n. A straightforward computation shows the following lemma which will be of later use.

LEMMA 1. *Let $f\colon \Delta \to \mathbb{R}$ and put $f_1(x, y) \equiv f(x, 0)$, $f_2(x, y) \equiv (0, y)$ for all $(x, y) \in \Delta$. Then, we have for each $(x, y) \in \Delta$*

$$(L_n f_1)(x, y) = (L_n f)(x, 0) \qquad \text{and} \qquad (L_n f_2)(x, y) = (L_n f)(0, y).$$

Remark 1. Note that with $g(x) = f(x, 0)$ and $h(y) = f(0, y)$ we have

$$(L_n f)(x, 0) = (L_n^{[1]} g)(x) \qquad \text{and}$$
$$(L_n f)(0, y) = (L_n^{[1]} h)(y) \qquad ((x, y) \in \Delta),$$

where $L_n^{[1]}$ denotes the one-dimensional BBH operator (1).

In the present section we study the moments of the BBH operators. Instead of the monomials $x^p y^q$ we consider the functions

$$g_{p,q}(x, y) = (1+x)^p (1+y)^q \qquad (p, q = 0, 1, 2, \ldots) \tag{12}$$

which are more suitable for the operators L_n. The first step is to express $L_n g_{p,q}$ as a certain double integral.

PROPOSITION 1. *For each* $(x, y) \in \Delta$, *we have in the case* $p, q \in \mathbb{N}$

$$\begin{aligned}(L_n g_{p,q})(x, y) &= (-1)^{p+q} \frac{(n+1)^{p+q}}{(p-1)!\,(q-1)!} \frac{(1+x)(1+y)}{(1-xy)^2} \\ &\quad \times \int_0^{\log(1+x)/(x(1+y))} \int_0^{\log(1+y)/(y(1+x))} \log^{p-1} \frac{(1+x)\,e^{-v_1} - x(1+y)}{1-xy} \\ &\quad \times \log^{q-1} \frac{(1+y)\,e^{-v_2} - y(1+x)}{1-xy} \left(\frac{e^{-v_1-v_2} - xy}{1-xy}\right)^n \\ &\quad \times e^{-v_1-v_2}\,dv_2\,dv_1, \end{aligned} \tag{13}$$

and in the case $p \in \mathbb{N}$, $q = 0$

$$\begin{aligned}(L_n g_{p,0})(x, y) = (-1)^{p-1} \frac{(n+1)^p}{(p-1)!} (1+x) \\ \times \int_0^{\log(1+x)/x} \log^{p-1} [(1+x)\,e^{-v} - x]\,e^{-(n+1)v}\,dv. \end{aligned} \tag{14}$$

The correspondent expression for the case $p = 0$, $q \in \mathbb{N}$ *is completely symmetric to Formula* (14).

Remark 2. Note that, for $(x, y) \in \Delta$, we have

$$\frac{1+x}{x(1+y)} = 1 + \frac{1-xy}{x(1+y)} > 1 \qquad \text{and} \qquad \frac{1+y}{y(1+x)} = 1 + \frac{1-xy}{y(1+x)} > 1.$$

Therefore, the integration domain in (13) is a proper rectangle in the first quadrant depending only on (x, y).

Remark 3. Since, for $a, b \geqslant 0$, not both equal to zero, we have

$$\int_a^\infty \int_b^\infty t_1^{p-1} t_2^{q-1} e^{-t_1-t_2} \left\{ \frac{1-xy}{(1+x)(1+y)} \left[e^{-t_1} + \frac{x(1+y)}{1-xy} \right] \left[e^{-t_2} + \frac{y(1+x)}{1-xy} \right] - \frac{xy}{(1-xy)} \right\}^n dt_2\,dt_1 = O(r^n) \qquad (n \to \infty)$$

with some positive constant $r < 1$, depending on x, y, a, b, the proof of Proposition 1 shows that the integration domain in (13) may be replaced by any smaller rectangle $[0, R_1] \times [0, R_2]$ with $0 < R_1 < \log(1+x)/(x(1+y))$ and $0 < R_2 < \log(1+y)/(y(1+x))$ producing an error of magnitude $O(e^{-\gamma n})$ with $\gamma > 0$ as $n \to \infty$.

The next proposition represents $L_n g_{p,q}$ in terms of a Laplace integral.

PROPOSITION 2. *For $(x, y) \in \Delta$ and $p, q \in \mathbb{N}$, there are positive numbers γ, δ independent on n such that*

$$(L_n g_{p,q})(x, y) = (n+1)^{p+q} \int_0^\delta e^{-(n+1)w} F(w)\, dw + O(e^{-\gamma n}) \qquad (n \to \infty), \tag{15}$$

where F is defined as

$$F(w) = (-1)^{p+q+1} \sum_{m=p-1}^{\infty} \sum_{\ell=q-1}^{\infty} S_m^{p-1} S_\ell^{q-1} \frac{(1+x)^{m+1}(1+y)^{\ell+1}}{(1-xy)^{m+\ell+1}} \times \sum_{\mu=m}^{\infty} \sum_{\lambda=\ell}^{\infty} \frac{1}{(\mu+\lambda+1)!} \log^{\mu+\lambda+1}[(1-xy)\, e^{-w} + xy]. \tag{16}$$

The quantities S_k^m and σ_k^m denote the Stirling numbers of the first, resp. second, kind. Recall that the Stirling numbers are defined by

$$x^{\underline{n}} = \sum_{k=0}^{n} S_n^k x^k \qquad \text{and} \qquad x^n = \sum_{k=0}^{n} \sigma_n^k x^{\underline{k}} \qquad (n \in \mathbb{N}_0),$$

where $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$, $x^{\underline{0}} = 1$ is the falling factorial.

For the proof of Proposition 2 we need the following preliminary lemma.

LEMMA 2. *For $m = 0, 1, 2, \ldots$, we have the power series expansions*

$$\log^m(1+x) = m! \sum_{k=m}^{\infty} S_k^m \frac{x^k}{k!} \qquad (|x| < 1)$$

and

$$(e^x - 1)^m = m! \sum_{k=m}^{\infty} \sigma_k^m \frac{x^k}{k!} \qquad (x \in \mathbb{R}).$$

For a proof see, e.g., [17, Eq. (4), p. 146 and Eq. (5), p. 202].

Moreover, we note the "orthogonality"-relation for the Stirling-numbers (see, e.g., [17, p. 182, Eq. (1)], resp. [17, p. 183, Eq. (2)]) which will be of later use.

LEMMA 3. *For $m, n = 0, 1, 2, \ldots$, with $m \leqslant n$ we have*

$$\sum_{k=m}^{n} \sigma_k^m \cdot S_n^k = \sum_{k=m}^{n} S_k^m \cdot \sigma_n^k = \begin{cases} 1, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

The next proposition gives the asymptotic expansion for $L_n g_{p,q}$ as n tends to infinity.

PROPOSITION 3. *For each $(x, y) \in \Delta$, the complete asymptotic expansion for $L_n g_{p,q}$ as $n \to \infty$ is*

(1) *in the case $p, q \in \mathbb{N}$*

$$\begin{aligned}(L_n g_{p,q})(x, y) &\sim (1+x)^p (1+y)^q + \sum_{k=1}^{\infty} \frac{(-1)^k}{(n+1)^k} \\ &\quad \times \sum_{j=0}^{k} \sum_{\mu+\lambda \leqslant j} \sum_{m=0}^{\mu} \sum_{\ell=0}^{\lambda} (1+x)^{m+p} (1+y)^{\ell+q} (1-xy)^{j-m-\ell} \\ &\quad \times S_{p-1+m}^{p-1} \sigma_{p-1+\mu}^{p-1+m} S_{q-1+\ell}^{q-1} \sigma_{q-1+\lambda}^{q-1+\ell} S_{p+q-1+j}^{p+q-1+\mu+\lambda} \sigma_{p+q-1+k}^{p+q-1+j}, \end{aligned} \tag{17}$$

(2) *in the case $p \in \mathbb{N}$, $q = 0$*

$$(L_n g_{p,0})(x, y) \sim (1+x)^p + \sum_{k=1}^{\infty} \frac{(-1)^k}{(n+1)^k} \sum_{m=0}^{k} S_{p-1+m}^{p-1} \sigma_{p-1+k}^{p-1+m} (1+x)^{m+p}.$$

The third case $p = 0$, $q \in \mathbb{N}$ runs completely symmetric to the second case.

Now we apply the following general approximation theorem [6, Theorem A] giving the complete asymptotic expansion for a sequence of positive linear operators in terms of their central moments.

THEOREM A. *Let $r \in \mathbb{N}$ and let $G \subset \mathbb{R}^2$. For $(x, y) \in G$, let $V_n \colon K^{[2r]}(x, y) \to C(G)$ $(n = 1, 2, \ldots)$ be a sequence of positive linear operators. Assume that the operators V_n are applicable to all polynomials of degree $\leqslant 2r+2$ and that*

$$V_n(((t_1 - x)^2 + (t_2 - y)^2)^s; x, y) = O(n^{-s}) \qquad (n \to \infty) \tag{18}$$

for $s = r$ and $s = r+1$. Then, we have, for each $f \in K^{[2r]}(x, y)$,

$$\begin{aligned}(V_n f)(x, y) &= \sum_{s=0}^{2r} \frac{1}{s!} \sum_{i+j=s} \binom{s}{i,\, j} \frac{\partial^s}{\partial x^i \, \partial y^j} f(x, y) \\ &\quad \times V_n((t_1 - x)^i (t_2 - y)^j; x, y) + o(n^{-r}) \qquad (n \to \infty). \end{aligned} \tag{19}$$

In order to obtain the complete asymptotic expansion (5) for the BBH operators we have to show that the operators L_n satisfy the assumptions of Theorem A with $G = \Delta$.

It remains to check condition (18), that is, we have to show that

$$L_n((t_1-x)^{2p}\,(t_2-y)^{2q};\,x,\,y)=O(n^{-s}) \qquad (n\to\infty)$$

for all $p, q\geqslant 0$ with $p+q=s$ ($s=0, 1, 2, ...$). In the following proposition we shall prove a slightly sharper result.

PROPOSITION 4. *For each $(x, y)\in\Delta$ and $P, Q=0, 1, 2, ...$, we have*

$$L_n((t_1-x)^{P}\,(t_2-y)^{Q};\,x,\,y)=O(n^{-\lfloor(P+Q+1)/2\rfloor}) \qquad (n\to\infty). \tag{20}$$

For the proof of Proposition 4 we shall need some further properties of the Stirling numbers [17, p. 151, Eq. (5)], resp. [17, p. 171, Eq. (7)].

LEMMA 4. *For k with $1\leqslant k\leqslant n$ the Stirling numbers of the first, resp. second, kind possess the representation*

$$S_n^{n-k}=C_{k,0}\binom{n}{2k}+\cdots+C_{k,k-1}\binom{n}{k+1}$$

and

$$\sigma_n^{n-k}=\bar{C}_{k,0}\binom{n}{2k}+\cdots+\bar{C}_{k,k-1}\binom{n}{k+1}.$$

The coefficients $C_{k,\ell}$ and $\bar{C}_{k,\ell}$ are independent on n and satisfy certain partial difference equations whose general solutions are unknown [17, p. 150]. Some closed expressions for $C_{k,\ell}$ and $\bar{C}_{k,\ell}$ can be found in [3] or [23].

Moreover, we mention the well-known expression

$$\sigma_n^k=\frac{(-1)^k}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}i^n$$

which implies, for all $k\in\mathbb{N}$,

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}i^n=(-1)^k\,k!\,\sigma_n^k=0 \qquad (n=0, ..., k-1) \tag{21}$$

with the convention $0^0=1$.

4. THE PROOFS

Proof of Proposition 1. We have, for $(x, y) \in \Delta$,

$$(L_n g_{p,q})(x, y) = \sum_{k+\ell \leqslant n} \frac{(n+1)^{p+q}}{(n-k+1)^p\,(n-\ell+1)^q} \binom{n}{k, \ell} \times \left(\frac{x}{1+x}\right)^k \left(\frac{y}{1+y}\right)^\ell \left(\frac{1-xy}{(1+x)(1+y)}\right)^{n-k-\ell}.$$

Taking advantage of the identity

$$z^{-p} = \frac{1}{(p-1)!} \int_0^\infty t^{p-1} e^{-tz}\, dt$$

for all $z > 0$ and $p \in \mathbb{N}$ we obtain for $p, q \in \mathbb{N}$

$$\begin{aligned}(L_n g_{p,q})(x, y) &= \frac{(n+1)^{p+q}}{(p-1)!\,(q-1)!} \int_0^\infty \int_0^\infty t_1^{p-1} t_2^{q-1} e^{-(n+1)(t_1+t_2)} \\ &\quad \times \sum_{k+\ell \leqslant n} \binom{n}{k, l} \left(\frac{xe^{t_1}}{1+x}\right)^k \left(\frac{ye^{t_2}}{1+y}\right)^\ell \left(\frac{1-xy}{(1+x)(1+y)}\right)^{n-k-\ell} dt_2\, dt_1 \end{aligned} \tag{22}$$

Application of the binomial theorem yields

$$\begin{aligned} &e^{-n(t_1+t_2)} \sum_{k+\ell \leqslant n} \binom{n}{k, \ell} \left(\frac{xe^{t_1}}{1+x}\right)^k \left(\frac{ye^{t_2}}{1+y}\right)^\ell \left(\frac{1-xy}{(1+x)(1+y)}\right)^{n-k-\ell} \\ &= \left\{\frac{1-xy}{(1+x)(1+y)} \left[e^{-t_1} + \frac{x(1+y)}{1-xy}\right] \left[e^{-t_2} + \frac{y(1+x)}{1-xy}\right] - \frac{xy}{(1-xy)}\right\}^n. \end{aligned}$$

Inserting this in (22) gives

$$\begin{aligned}(L_n g_{p,q})(x, y) &= \frac{(n+1)^{p+q}}{(p-1)!\,(q-1)!} \int_0^\infty \int_0^\infty t_1^{p-1} t_2^{q-1} e^{-t_1-t_2} \left\{\frac{1-xy}{(1+x)(1+y)}\right. \\ &\quad \times \left.\left[e^{-t_1} + \frac{x(1+y)}{1-xy}\right] \left[e^{-t_2} + \frac{y(1+x)}{1-xy}\right] - \frac{xy}{(1-xy)}\right\}^n dt_2\, dt_1. \end{aligned}$$

If we change the variables according to

$$t_1 = -\log \frac{(1+x)\, e^{-v_1} - x(1+y)}{1-xy} \qquad \text{and}$$

$$t_2 = -\log \frac{(1+y)\, e^{-v_2} - y(1+x)}{1-xy},$$

we get (13).

Observing that for $p, q \in \mathbb{N}$ there holds $g_{p,0}(x, y) = \lim_{y \to 0} g_{p,q}(x, y)$ and $g_{0,q}(x, y) = \lim_{x \to 0} g_{p,q}(x, y)$ the remaining cases follow, by Lemma 1. This completes the proof of Proposition 1.

Proof of Proposition 2. According to Remark 2 the integration domain in (13) is a rectangle in the first quadrant depending only on (x, y). A rotation of the rectangle by $\pi/4$ around the origin, i.e., the change of variable

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

gives, with regard to Remark 3,

$$\begin{aligned}
&(L_n g_{p,q})(x, y) \\
&\quad = (-1)^{p+q} \frac{(n+1)^{p+q}}{(p-1)!\,(q-1)!} \frac{(1+x)(1+y)}{(1-xy)^2} \\
&\qquad \times \int_0^{\varepsilon} \int_{-w_2}^{w_2} \log^{p-1} \frac{(1+x)\, e^{-(w_1+w_2)/2} - x(1+y)}{1-xy} \\
&\qquad \times \log^{q-1} \frac{(1+y)\, e^{-(-w_1+w_2)/2} - y(1+x)}{1-xy} \left(\frac{e^{-w_2} - xy}{1-xy} \right)^n e^{-w_2} \frac{1}{2}\, dw_1\, dw_2 \\
&\qquad + O(e^{-\gamma n}) \qquad (n \to \infty)
\end{aligned} \tag{23}$$

for arbitrary small $\varepsilon > 0$ with a constant $\gamma > 0$ depending only on (x, y) and ε.

A further change of variable replacing w_1 by $2w_1 w_2 - w_2$ in the inner integral leads to

$$(L_n g_{p,q})(x,y) = (-1)^{p+q} \frac{(n+1)^{p+q}}{(p-1)!\,(q-1)!} \frac{(1+x)(1+y)}{(1-xy)^2}$$
$$\times \int_0^{\varepsilon} \int_0^1 \log^{p-1} \frac{(1+x)\,e^{-w_1 w_2} - x(1+y)}{1-xy}$$
$$\times \log^{q-1} \frac{(1+y)\,e^{-w_2(1-w_1)} - y(1+x)}{1-xy}\, dw_1$$
$$\times \left(\frac{e^{-w_2} - xy}{1-xy} \right)^n w_2 e^{-w_2}\, dw_2 + O(e^{-\gamma n}) \qquad (n \to \infty). \tag{24}$$

Without loss of generality we can assume that ε in (24) is so small that

$$\left| \frac{1+x}{1-xy} (e^{-w_1 w_2} - 1) \right| < 1$$

and

$$\left| \frac{1+y}{1-xy} (e^{-w_2(1-w_1)} - 1) \right| < 1$$

for all $w_1 \in [0, 1]$ and $w_2 \in [0, \varepsilon]$. Then, we have, by Lemma 2,

$$\log^{p-1} \frac{(1+x)\,e^{-w_1 w_2} - x(1+y)}{1-xy}$$
$$= \log^{p-1} \left(1 + \frac{1+x}{1-xy} (e^{-w_1 w_2} - 1) \right)$$
$$= (p-1)! \sum_{m=p-1}^{\infty} S_m^{p-1} \left(\frac{1+x}{1-xy} \right)^m \sum_{\mu=m}^{\infty} \frac{1}{\mu!} \sigma_\mu^m (-w_1 w_2)^\mu.$$

Inserting this and the analogous expansions for the $\log^{q-1}$-term in Eq. (24) we obtain

$$(L_n g_{p,q})(x,y) = (-1)^{p+q} (n+1)^{p+q} \frac{(1+x)(1+y)}{(1-xy)^2}$$
$$\times \sum_{m=p-1}^{\infty} \sum_{\ell=q-1}^{\infty} S_m^{p-1} S_\ell^{q-1} \left(\frac{1+x}{1-xy} \right)^m \left(\frac{1+y}{1-xy} \right)^\ell$$
$$\times \sum_{\mu=m}^{\infty} \sum_{\lambda=\ell}^{\infty} \frac{(-1)^{\mu+\lambda}}{\mu!\,\lambda!} \sigma_\mu^m \sigma_\lambda^\ell$$

$$\times \int_0^{\varepsilon} \int_0^1 w_1^{\mu}(1-w_1)^{\lambda}\, dw_1 \left(\frac{e^{-w_2}-xy}{1-xy}\right)^n w_2^{\mu+\lambda+1} e^{-w_2}\, dw_2$$

$$+O(e^{-\gamma n}) \qquad (n \to \infty).$$

A last change of variable $w_2 = -\log[(1-xy)\, e^{-w} + xy]$, and noting that the inner integral is the Beta-function $B(\mu+1, \lambda+1) = \mu!\, \lambda!/((\mu+\lambda+1)!)$ yields, finally,

$$(L_n\, g_{p,q})(x, y) = (-1)^{p+q+1}\, (n+1)^{p+q} \sum_{m=p-1}^{\infty} \sum_{\ell=q-1}^{\infty} S_m^{p-1} S_{\ell}^{q-1}$$

$$\times \frac{(1+x)^{m+1}\, (1+y)^{\ell+1}}{(1-xy)^{m+\ell+1}} \sum_{\mu=m}^{\infty} \sum_{\lambda=\ell}^{\infty} \frac{1}{(\mu+\lambda+1)!}\, \sigma_{\mu}^{m} \sigma_{\lambda}^{\ell}$$

$$\times \int_0^{\delta} e^{-(n+1)\, w} \log^{\mu+\lambda+1} [(1-xy)\, e^{-w} + xy]\, dw$$

$$+O(e^{-\gamma n}) \qquad (n \to \infty)$$

with $\delta = -\log((e^{-\varepsilon} - xy)/(1-xy))$. Note that $\delta > 0$ if ε is sufficiently small. This completes the proof of Proposition 2.

Proof of Proposition 3. We start with the case $p, q \in \mathbb{N}$. By Proposition 2, $(L_n\, g_{p,q})(x, y)$ is, essentially, the Laplace transform of the truncated function

$$\tilde{F}(w) = \begin{cases} F(w) & (|w| \leqslant \delta), \\ 0 & (|w| > \delta), \end{cases}$$

with F as defined in (16). In order to derive an asymptotic expansion for $(L_n\, g_{p,q})(x, y)$ we study the behaviour of the Laplace integral in (15). Obviously, F is analytic in a neighborhood of the origin $w = 0$. We proceed in deriving the power series expansion of F.

Application of Lemma 2 yields

$$\log^{\mu+\lambda+1} [(1-xy)\, e^{-w} + xy]$$

$$= \log^{\mu+\lambda+1} [1 + (1-xy)(e^{-w} - 1)]$$

$$= (\mu+\lambda+1)! \sum_{\rho=\mu+\lambda+1}^{\infty} \frac{(-w)^{\rho}}{\rho!} \sum_{\tau=\mu+\lambda+1}^{\rho} S_{\tau}^{\mu+\lambda+1} \sigma_{\rho}^{\tau} (1-xy)^{\tau}$$

in $|w| < \delta$. Therefore, by (16), we have

$$F(w) = (-1)^{p+q+1} \sum_{\mu=p-1}^{\infty} \sum_{\lambda=q-1}^{\infty} \sum_{m=p-1}^{\mu} \sum_{\ell=q-1}^{\lambda} S_m^{p-1} S_\ell^{q-1} \sigma_\mu^m \sigma_\lambda^\ell$$

$$\times \frac{(1+x)^{m+1} (1+y)^{\ell+1}}{(1-xy)^{m+\ell+1}} \sum_{\rho=\mu+\lambda+1}^{\infty} \frac{(-w)^\rho}{\rho!}$$

$$\times \sum_{\tau=\mu+\lambda+1}^{\rho} S_\tau^{\mu+\lambda+1} \sigma_\rho^\tau (1-xy)^\tau$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k w^{k+p+q-1}}{(k+p+q-1)!} \sum_{\rho+\mu+\lambda=k} \sum_{m=0}^{\mu} \sum_{\ell=0}^{\lambda} S_{m+p-1}^{p-1}$$

$$\times S_{\ell+q-1}^{q-1} \sigma_{\mu+p-1}^{m+p-1} \sigma_{\lambda+q-1}^{\ell+q-1}$$

$$\times \frac{(1+x)^{m+p} (1+y)^{\ell+q}}{(1-xy)^{m+\ell}} \sum_{\tau=0}^{\rho} S_{\tau+\mu+\lambda+p+q-1}^{\mu+\lambda+p+q-1}$$

$$\times \sigma_{k+p+q-1}^{\tau+\mu+\lambda+p+q-1} (1-xy)^{\tau+\mu+\lambda}.$$

Thus, we can apply Watson's lemma (see, e.g., [20, p. 106f] or [1, Lemma 1]) which states that

$$\int_0^\delta e^{-sw} F(w)\, dw \sim \sum_{k=0}^{\infty} k!\, a_k s^{-k-1} \qquad (s \to \infty),$$

provided $F(w) = \sum_{k=0}^{\infty} a_k w^k$. We remark that this is valid even if s is complex as $\Re(s) \to +\infty$.

By Proposition 2, we obtain

$$(L_n g_{p,q})(x, y) \sim \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+1)^k} \sum_{\rho+\mu+\lambda=k} \sum_{m=0}^{\mu} \sum_{\ell=0}^{\lambda} S_{m+p-1}^{p-1}$$

$$\times S_{\ell+q-1}^{q-1} \sigma_{\mu+p-1}^{m+p-1} \sigma_{\lambda+q-1}^{\ell+q-1} \frac{(1+x)^{m+p} (1+y)^{\ell+q}}{(1-xy)^{m+\ell}}$$

$$\times \sum_{\tau=0}^{\rho} S_{\tau+\mu+\lambda+p+q-1}^{\mu+\lambda+p+q-1} \sigma_{k+p+q-1}^{\tau+\mu+\lambda+p+q-1} (1-xy)^{\tau+\mu+\lambda}$$

as $n \to \infty$ which yields Eq. (17).

The cases (2) and (3) can be deduced directly from Proposition 1. We obtain them, in an alternative way, from the first case. By Lemma 1, we put $y=0$ in Eq. (17) with arbitrary $q \in \mathbb{N}$ in order to obtain

$$
\begin{aligned}
&(L_n g_{p,0})(x, y)\\
&\quad = (L_n g_{p,q})(x, 0)\\
&\quad \sim (1+x)^p + \sum_{k=1}^{\infty} \frac{(-1)^k}{(n+1)^k} \sum_{j=0}^{k} \sum_{\mu+\lambda \leqslant j} \sum_{m=0}^{\mu} (1+x)^{m+p}\\
&\qquad \times S_{p-1+m}^{p-1} \sigma_{p-1+\mu}^{p-1+m} S_{p+q-1+j}^{p+q-1+\mu+\lambda} \sigma_{p+q-1+k}^{p+q-1+j} \sum_{\ell=0}^{\lambda} S_{q-1+\ell}^{q-1} \sigma_{q-1+\lambda}^{q-1+\ell}\\
&\quad = (1+x)^p + \sum_{k=1}^{\infty} \frac{(-1)^k}{(n+1)^k} \sum_{m=0}^{k} (1+x)^{m+p} S_{p-1+m}^{p-1} \sigma_{p-1+k}^{p-1+m},
\end{aligned}
$$

where we used Lemma 3 twice. Likewise, case (3) follows from Eq. (17).

Proof of Proposition 4. The case $P+Q=0$ is obvious. Now let $P+Q>0$. Application of the binomial theorem yields

$$
\begin{aligned}
&L_n((t_1-x)^P (t_2-y)^Q; x, y)\\
&\quad = \sum_{p=0}^{P} \sum_{q=0}^{Q} (-1)^{P+Q-p-q} \binom{P}{p} \binom{Q}{q} (1+x)^{P-p} (1+y)^{Q-q} (L_n g_{p,q})(x, y).
\end{aligned}
$$

We split the sum and obtain, by Proposition 3,

$$
(-1)^{P+Q} L_n((t_1-x)^P (t_2-y)^Q; x, y) \sim \sum_{k=1}^{\infty} \frac{(-1)^k}{(n+1)^k} (\Sigma_1+\Sigma_2+\Sigma_3), \tag{25}
$$

say, with

$$
\begin{aligned}
\Sigma_1 &= \sum_{j=0}^{k} \sum_{\mu+\lambda \leqslant j} \sum_{m=0}^{\mu} \sum_{\ell=0}^{\lambda} (1+x)^{P+m} (1+y)^{Q+\ell} (1-xy)^{j-m-\ell}\\
&\quad \times \sum_{p=1}^{P} \sum_{q=1}^{Q} (-1)^{p+q}\\
&\quad \times \binom{P}{p} \binom{Q}{q} S_{p-1+m}^{p-1} S_{q-1+\ell}^{q-1} \sigma_{p-1+\mu}^{p-1+m} \sigma_{q-1+\lambda}^{q-1+\ell} S_{p+q-1+j}^{p+q-1+\mu+\lambda} \sigma_{p+q-1+k}^{p+q-1+j},\\
\Sigma_2 &= (1+y)^Q \sum_{m=0}^{k} (1+x)^{P+m} \sum_{p=1}^{P} (-1)^p \binom{P}{p} S_{p-1+m}^{p-1} \sigma_{p-1+k}^{p-1+m},\\
\Sigma_3 &= (1+x)^P \sum_{\ell=0}^{k} (1+y)^{Q+\ell} \sum_{q=1}^{Q} (-1)^q \binom{Q}{q} S_{q-1+\ell}^{q-1} \sigma_{q-1+k}^{q-1+\ell},
\end{aligned}
\tag{26}
$$

where the sums are to be read as 0 if $P=0$ or $Q=0$, respectively. We shall prove that $\Sigma_1+\Sigma_2+\Sigma_3=0$ if $2k<P+Q$.

Let us first consider Σ_1. For fixed $k, j, \mu, \lambda, m, \ell$, the term

$$S_{p-1+m}^{p-1} S_{q-1+\ell}^{q-1} \sigma_{p-1+\mu}^{p-1+m} \sigma_{q-1+\lambda}^{q-1+\ell} S_{p+q-1+j}^{p+q-1+\mu+\lambda} \sigma_{p+q-1+k}^{p+q-1+j}$$

occurring in Σ_1 is, by Lemma 4, a polynomial in p and q of order $\leqslant 2k$ which we shall denote by $T(p,q)=T(k, j, \mu, \lambda, m, \ell; p, q)$.

Assume $2k<P+Q$. Then, we have, by Eq. (21),

$$\begin{aligned} &\sum_{p=1}^{P} \sum_{q=1}^{Q} (-1)^{p+q} \binom{P}{p} \binom{Q}{q} T(p,q) \\ &\quad = -\sum_{p=1}^{P} (-1)^p \binom{P}{p} T(p,0) - \sum_{q=1}^{Q} (-1)^q \binom{Q}{q} T(0,q) - T(0,0). \end{aligned} \tag{27}$$

By Lemma 4, S_{p-1+m}^{p-1} is a polynomial in p which vanishes for $p=0$ if $m>0$ and takes the value 1 for $p=0$ if $m=0$. Likewise $\sigma_{p-1+\mu}^{p-1+m}$ is a polynomial in p which vanishes for $p=0$ if $m=0$, $\mu>0$ and takes the value 1 for $p=0$ if $m=\mu=0$.

With the same argument $S_{q-1+\ell}^{q-1}\sigma_{q-1+\lambda}^{q-1+\ell}$ is a polynomial in q which vanishes for $q=0$ if $\ell>0$ or $\ell=0$, $\lambda>0$ and takes the value 1 for $q=0$ if $\ell=\lambda=0$.

Therefore,

$$\sum_{p=1}^{P} (-1)^p \binom{P}{p} T(p,0)=0$$

unless $\ell=\lambda=0$ in which case we have

$$\begin{aligned} &\sum_{j=0}^{k} \sum_{\mu+\lambda\leqslant j} \sum_{m=0}^{\mu} \sum_{\ell=0}^{\lambda} (1+x)^{P+m} (1+y)^{Q+\ell} (1-xy)^{j-m-\ell} \\ &\qquad \times \sum_{p=1}^{P} (-1)^p \binom{P}{p} T(k, j, \mu, \lambda, m, \ell; p, 0) \\ &\quad = (1+y)^Q \sum_{j=0}^{k} (1+x)^{P+j} \sum_{p=1}^{P} (-1)^p \binom{P}{p} S_{p-1+j}^{p-1} \sigma_{p-1+k}^{p-1+j} = \Sigma_2, \end{aligned} \tag{28}$$

where we used Lemma 3. In a completely analogous fashion one shows

$$\begin{aligned} &\sum_{j=0}^{k} \sum_{\mu+\lambda\leqslant j} \sum_{m=0}^{\mu} \sum_{\ell=0}^{\lambda} (1+x)^{P+m} (1+y)^{Q+\ell} (1-xy)^{j-m-\ell} \\ &\qquad \times \sum_{q=1}^{Q} (-1)^q \binom{Q}{q} T(k, j, \mu, \lambda, m, \ell; 0, q) = \Sigma_3. \end{aligned} \tag{29}$$

Furthermore, $T(0,0)=0$ if not $m=\mu=\ell=\lambda=0$. In this case T reduces to $S^{p+q-1}_{p+q-1+j}\sigma^{p+q-1+j}_{p+q-1+k}$ which is a polynomial in p, q vanishing for $p=q=0$ if $j>0$ or if $j=0$ and $k>0$. Therefore, we have

$$\sum_{j=0}^{k} \sum_{\mu+\lambda\leqslant j} \sum_{m=0}^{\mu} \sum_{\ell=0}^{\lambda} (1+x)^{P+m}(1+y)^{Q+\ell}(1-xy)^{j-m-\ell} \times T(k, j, \mu, \lambda, m, \ell; 0, 0)=0 \qquad (30)$$

for $k>0$. Combining Eqs. (28), (29), and (30) with (27) we obtain $\Sigma_1 = -\Sigma_2 - \Sigma_3$ in Eq. (25).

This completes the proof of Proposition 4.

Proof of Theorem 1. Theorem 1 follows from Proposition 3, Theorem A, Proposition 4, Eqs. (25), (26) and the observation that, for each linear function f,

$$(L_n f)(x, y) = f(x, y) + O(e^{-qn}) \qquad (n \to \infty)$$

with some $q>0$.

Proof of Corollary 2. By Remark 1, Corollary 2 is an immediate consequence of Theorem 1.

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